

2^k Factorial Design

- Each factor has two levels (often labeled + and -)
- Very useful design for preliminary analysis
- Can “weed out” unimportant factors
- Also allows initial study of interactions
- For general two-factor factorial model

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}$$

- Have $1 + (a - 1) + (b - 1) + (a - 1)(b - 1)$ parameters
- 2² factorial has only four parameters

$$(\alpha\beta)_{11} = -\alpha_2 \quad (\alpha\beta)_{12} = -(\alpha\beta)_{22} \quad (\alpha\beta)_{21} = -\beta_2$$

- Can study 2 factors through repeated 2² designs
 - Select different levels of each factor

2^k Factorial Models

Design of Experiments - Montgomery
Chapter 6

Design for 2² Factorial

- Label the levels of factors *A* and *B* using + and -
- Factors need not be on numeric scale
- There are 4 experimental combinations labeled

<i>A</i>	<i>B</i>	Symbol
-	-	(1)
+	-	a
-	+	b
+	+	ab

- Can express combination in terms of model params

$$\begin{aligned} ab = (+, +) : E(y) &= \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\ a = (+, -) : E(y) &= \mu + \alpha_2 + \beta_1 + (\alpha\beta)_{21} \\ b = (-, +) : E(y) &= \mu + \alpha_1 + \beta_2 + (\alpha\beta)_{12} \\ (1) = (-, -) : E(y) &= \mu + \alpha_1 + \beta_1 + (\alpha\beta)_{11} \end{aligned}$$

$$\begin{aligned} ab = (+, +) : E(y) &= \mu + \alpha_2 + \beta_2 + (\alpha\beta)_{22} \\ a = (+, -) : E(y) &= \mu + \alpha_2 - \beta_2 - (\alpha\beta)_{22} \\ b = (-, +) : E(y) &= \mu - \alpha_2 + \beta_2 - (\alpha\beta)_{22} \\ (1) = (-, -) : E(y) &= \mu - \alpha_2 - \beta_2 + (\alpha\beta)_{22} \end{aligned}$$

Estimating Model Parameters

- Have four equations and four unknowns

$$\begin{aligned} \hat{\mu} &= \frac{ab + a + b + (1)}{4n} \\ \hat{\alpha}_2 &= .5\left(\frac{ab+a}{2n} - \frac{b+(1)}{2n}\right) = \frac{ab + a - b - (1)}{4n} \\ \hat{\beta}_2 &= .5\left(\frac{ab+b}{2n} - \frac{a+(1)}{2n}\right) = \frac{ab - a + b - (1)}{4n} \\ \widehat{(\alpha\beta)}_{22} &= .5\left(\frac{ab-b}{2n} - \frac{a-(1)}{2n}\right) = \frac{ab - a - b + (1)}{4n} \end{aligned}$$

where *n* represents the number of replications of each combination and *ab*, *b*, etc represent the sample sums of each combination.

- Main effect of *A* defined as $2\hat{\alpha}_2$
- Main effect of *B* defined as $2\hat{\beta}_2$
- Interaction effect defined as $2(\hat{\alpha\beta})_{22}$
- Each effect function of ± 1 (combination sums)

Using Effects To Draw Inference

- Look at magnitude and direction of effects
- ANOVA then used to verify conclusions (Chapt 5)
- Effects and contrasts related as shown

Effect	(1)	a	b	ab
A	-1	+1	-1	+1
B	-1	-1	+1	+1
AB	1	-1	-1	+1

- AB coefficients simply product of A and B coeffs
- Sum of squares related to model effects

$$SS_{\text{Contrast}} = \frac{(\sum c_i y_i)^2}{n \sum c_i^2} \quad i = 1, 2, 3, 4$$

$$SS_A = \frac{(ab+a-b-(1))^2}{4n} = 4n\hat{\alpha}_2^2$$

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Regression / Response Surface

- Code each factor as ± 1 where +1 is high level
- x_1 associated with Factor 1 = A
- x_2 associated with Factor 2 = B
- Regression Model

$$y_{ijk} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon_{ijk}$$

- Can show

$$\beta_0 = \hat{\mu} \quad \beta_1 = \hat{\alpha}_2 \quad \beta_2 = \hat{\beta}_2$$

$$\beta_{12} = (\widehat{\alpha\beta})_{22}$$

- Parameters based on unit increase (not -1 to +1)
- Can use regression model to form contours
- No interaction \rightarrow linear relationship
- Can often see direction for improvement in response

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Hormone Example

Combination	Total
(1)	467
a	394
b	642
ab	571

$$\hat{\mu} = \frac{571+394+642+467}{4(6)} = 90.75$$

$$\hat{\alpha}_2 = \frac{571+394-642-467}{4(6)} = -6$$

$$\hat{\beta}_2 = \frac{571-394+642-467}{4(6)} = 14.6666$$

$$(\widehat{\alpha\beta})_{22} = \frac{571-394-642+467}{4(6)} = .08333$$

- $SS_A = 4(6)\hat{\alpha}_2^2 = 864$
- $SS_B = 4(6)\hat{\beta}_2^2 = 5162.67$
- $SS_{AB} = 4(6)\widehat{\alpha\beta}_{22}^2 = .17$
- Response Surface model

$$y = 90.75 + 14.67x_2 + \epsilon$$

To increase response, model says to increase level. Be careful of interpolation.

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2^3 Factorial Designs

- Eight combinations displayed as cube

A	B	C	Symbol
-	-	-	(1)
+	-	-	a
-	+	-	b
+	+	-	ab
-	-	+	c
+	-	+	ac
-	+	+	bc
+	+	+	abc

- A main effect is equal to

$$\frac{2\hat{\alpha}_2}{4n} = \frac{(a-(1))+(ab-b)+(ac-c)+(abc-bc)}{4n}$$

$$\frac{(a-(1))(b+(1))(c+(1))}{4n}$$

- The AB interaction is equal to

$$2(\widehat{\alpha\beta})_{22} = \frac{((abc-bc)+(ab-b))-((ac-c)+(a-(1)))}{4n}$$

$$\frac{(a-(1))(b-(1))(c+(1))}{4n}$$

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2³ Factorial

Contrast Coefficients							
Symbol	A	B	AB	C	AC	BC	ABC
(1)	-	-	+	-	+	+	-
a	+	-	-	-	-	+	+
b	-	+	-	-	+	-	+
ab	+	+	+	-	-	-	+
c	-	-	+	+	-	-	+
ac	+	-	-	+	+	-	-
bc	-	+	-	+	-	+	-
abc	+	+	+	+	+	+	+

$$\begin{aligned} \text{Contrast} &= \sum c_i y_i = (\pm 1) \pm a \pm b \pm ab \pm c \pm ac \pm bc \pm abc \\ \text{Parameter} &= \text{Contrast}/8n \\ \text{Effect} &= \text{Contrast}/4n = 2\text{Parameter} \end{aligned}$$

- $SS_{\text{Effect}} = (\text{Contrast})^2 / 8n$
- $SS_{\text{Effect}} = 2n(\text{Effect})^2$
- $\text{Var}(\text{Contrast}) = 8n\sigma^2$
- $\text{Var}(\text{Effect}) = \sigma^2 / 2n$

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General 2^k Factorial

- Contrast is sum of trt combinations (± 1 coeff)

$$\text{Var}(\text{Contrast}) = n2^k\sigma^2$$

- Effect is Contrast divided by $n2^{k-1}$

$$\text{Var}(\text{Effect}) = \sigma^2 / n2^{k-2}$$

- Can use variances to test significance of effect

Approximate 95% CI for Effect

$$\text{Effect} \pm 1.96\sqrt{\sigma^2 / n2^{k-2}}$$

- Look at effects whose intervals do not contain 0

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Yates Algorithm

- For general 2^k design
 - List data in standard order
 - Create k new columns in the following manner
 - The first 2^{k-1} entries are obtained by adding adjacent values
 - The last 2^{k-1} entries are obtained by subtracting adjacent values
 - Estimates of effects obtained by dividing k th column by $n2^{k-1}$
 - Sum of Squares obtained by squaring k th column and dividing by $n2^k$

Combination	y	1	2	3	Estimate
(1)	15				
a	10				
b	5				
ab	25				
c	15				
ac	25				
bc	10				
abc	5				

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Single Replicate In 2^k Design

- With only one replicate
 - Cannot estimate all interactions and σ^2
- Often assume 3 or higher interactions do not occur
- Pool estimates together for error
- Warning: may pool significant interaction
- Create Normal Probability Plot of Effects
- If negligible, effect will be $N(0, \sigma^2)$
- Will fall along straight line
- Significant effects will not lie along same line
- Use Yates Algorithm or SAS to create plot

23-11

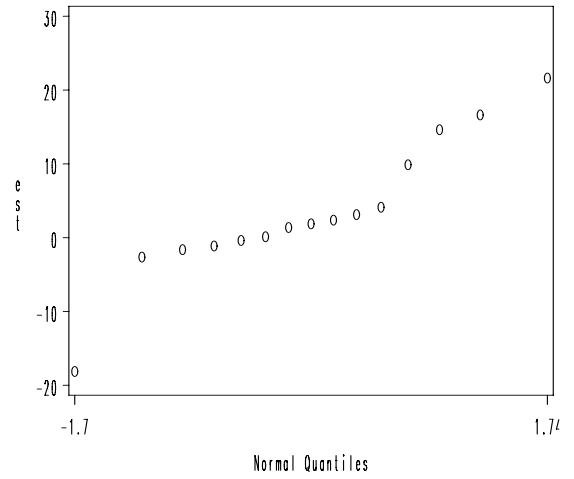
Example 6-2

```

data new;
input fact1 fact2 fact3 fact4 est;
cards;
 2 1 1 1 21.625
 1 2 1 1 3.125
 2 2 1 1 0.125
 1 1 2 1 9.875
 2 1 2 1 -18.125
 1 2 2 1 2.375
 2 2 2 1 1.875
 1 1 1 2 14.625
 2 1 1 2 16.625
 1 2 1 2 -0.375
 2 2 1 2 4.125
 1 1 2 2 -1.125
 2 1 2 2 -1.625
 1 2 2 2 -2.625
 2 2 2 2 1.375
;

proc univariate noprint;
qqplot est / normal;

```



```

data filter;
do D = -1 to 1 by 2; do C = -1 to 1 by 2;
do B = -1 to 1 by 2; do A = -1 to 1 by 2;
input y @@; output;
end; end; end; end;
cards;
45 71 48 65 68 60 80 65 43 100 45 104 75 86 70 96
;

data inter; /* Define Interaction Terms */
set filter;
AB=A*B; AC=A*C; AD=A*D; BC=B*C; BD=B*D; CD=C*D; ABC=AB*C; ABD=AB*D;
ACD=AC*D; BCD=BC*D; ABCD=ABC*D;

proc glm data=inter; /* GLM Proc to Obtain Effects */
class A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
estimate 'A' A 1 -1; estimate 'AC' AC 1 -1;

proc reg outest=effects data=inter; /* REG Proc to Obtain Effects */
model y=A B C D AB AC AD BC BD CD ABC ABD ACD BCD ABCD;
data effect2; set effects;
drop y intercept _RMSE_;
proc transpose data=effect2 out=effect3;
data effect4; set effect3; effect=col1*2;
proc sort data=effect4; by effect;
proc print data=effect4;
proc rank data=effect4 normal=blom;
var effect; ranks neff;

symbol1 v=circle;
proc gplot;
plot neff*effect=_NAME_;

```

