

Computing Standard Errors

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Simple Random Effects

$$y_{ij} = \mu + \tau_i + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, n \end{cases}$$

$$\tau_i \sim N(0, \sigma_\tau^2) \text{ and } \epsilon_{ij} \sim N(0, \sigma^2)$$

$\{\tau_i\}$ and $\{\epsilon_{ij}\}$ independent

$$\begin{aligned} \text{Var}(\bar{y}_{..}) &= \text{Var}(\mu) + \text{Var}(\bar{\tau}_{..}) + \text{Var}(\bar{\epsilon}_{..}) \\ &= 0 + \sigma_\tau^2/a + \sigma^2/an \\ &= (n\sigma_\tau^2 + \sigma^2)/an \end{aligned}$$

Since $E(\text{MS}_A) = n\sigma_\tau^2 + \sigma^2$, we use this mean square and the associated degrees of freedom associated when constructing a confidence interval or performing a hypothesis test.

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Two Factor Random Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, n \end{cases}$$

$$\tau_i \sim N(0, \sigma_\tau^2) \text{ and } \beta_j \sim N(0, \sigma_\beta^2)$$

$$(\tau\beta)_{ij} \sim N(0, \sigma_{\tau\beta}^2) \text{ and } \epsilon_{ijk} \sim N(0, \sigma^2)$$

$\{\tau_i\}$, $\{\beta_j\}$, $\{(\tau\beta)_{ij}\}$ and $\{\epsilon_{ijk}\}$ independent

$$\begin{aligned} \text{Var}(\bar{y}_{...}) &= \text{Var}(\mu + \bar{\tau}_{..} + \bar{\beta}_{..} + \overline{(\tau\beta)}_{..} + \bar{\epsilon}_{...}) \\ &= 0 + \sigma_\tau^2/a + \sigma_\beta^2/b + \sigma_{\tau\beta}^2/ab + \sigma^2/abn \\ &= (bn\sigma_\tau^2 + an\sigma_\beta^2 + n\sigma_{\tau\beta}^2 + \sigma^2)/abn \end{aligned}$$

In this case, there is no expected mean square equal the numerator. As a result, the combination $\text{MS}_A + \text{MS}_B - \text{MS}_{AB}$ is used to estimate the variance and Satterthwaite's degrees of freedom formula is used to approximate the degrees of freedom.

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Two Factor Mixed Effects Model

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, n \end{cases}$$

$$\begin{aligned} \sum \tau_i &= 0 & \text{and} & & \beta &\sim N(0, \sigma_\beta^2) \\ (\tau\beta)_{ij} &\sim N(0, (a-1)\sigma_{\tau\beta}^2/a) & \text{and} & & \sum_i (\tau\beta)_{ij} &= 0 \text{ for } \beta \text{ level } j \\ & & \epsilon_{ijk} &\sim N(0, \sigma^2) & & \end{aligned}$$

$$\begin{aligned} \text{Var}(\bar{y}_{i..}) &= \text{Var}(\mu + \tau_i + \bar{\beta}_{..} + \overline{(\tau\beta)}_{i..} + \bar{\epsilon}_{i..}) \\ &= 0 + \sigma_\tau^2/b + (a-1)\sigma_{\tau\beta}^2/ab + \sigma^2/bn \end{aligned}$$

The unrestricted model would not have this $\frac{a-1}{a}$ coefficient in front of the $\sigma_{\tau\beta}^2$. In either case, the estimate of the variance can only be written in the form $p_1\text{MS}_1 + p_2\text{MS}_2 + \dots + p_k\text{MS}_k$ where some of the p_i are different than ± 1 . The formula on page 536 can be expanded to approximate this situation. It is simply

$$df = \frac{(\sum p_i \text{MS}_i)^2}{\sum p_i^2 \text{MS}_i^2 / df_i}$$

$$\begin{aligned} \text{Var}(\bar{y}_{i..} - \bar{y}_{i'jk}) &= \text{Var}(\tau_i - \tau_{i'} + \overline{(\tau\beta)}_{i..} - \overline{(\tau\beta)}_{i'..} + \bar{\epsilon}_{i..} - \bar{\epsilon}_{i'..}) \\ &= 2\sigma_\tau^2/b + 2\sigma^2/bn \end{aligned}$$

In both the restricted and unrestricted models, the variance of the difference between two treatment means is the same. Here we would use MS_{AB} and its degrees of freedom when performing hypothesis tests.

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Split Plot Design

Will use unrestricted mixed model and look at both pooled and unpooled subplot error. Will also focus of RCBD in whole plot with no replication.

There are several comparisons that may be of interest.

- 1 Main effect in whole plot
- 2 Main effect in subplot
- 3 Interaction with i fixed
- 4 Interaction with k fixed

Pooled

$$y_{ijk} = \mu + B_j + A_i + AB_{ij} + C_k + AC_{ik} + \epsilon_{ijk} \quad \begin{cases} i = 1, 2, \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, c \end{cases}$$

$$\begin{aligned} \sum A_i &= 0 & \text{and} & & B &\sim N(0, \sigma_B^2) \\ AB_{ij} &\sim N(0, \sigma_{AB}^2) & \text{and} & & \epsilon_{ijk} &\sim N(0, \sigma^2) \\ \sum C_k &= 0 & \text{and} & & \sum \sum AC_{ik} &= 0 \end{aligned}$$

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$$\begin{aligned} \bar{y}_{i..} &= \mu + \bar{B} + A_i + \overline{AB}_{i.} + \bar{C} + \overline{AC}_{i.} + \bar{\epsilon}_{i.} \\ \bar{y}_{.k} &= \mu + \bar{B} + \bar{A} + \overline{AB}_{.k} + C_k + \overline{AC}_{.k} + \bar{\epsilon}_{.k} \\ \bar{y}_{i.k} &= \mu + \bar{B} + A_i + \overline{AB}_{i.} + C_k + AC_{ik} + \bar{\epsilon}_{i.k} \end{aligned}$$

- 1 Use MS_{AB} in calculations

$$\begin{aligned} \text{Var}(\bar{y}_{i.} - \bar{y}_{i'..}) &= \text{Var}(A_i - A_{i'} + \overline{AB}_{i.} - \overline{AB}_{i'..} + \bar{\epsilon}_{i.} - \bar{\epsilon}_{i'..}) \\ &= 2(\sigma_{AB}^2/b + \sigma^2/bc) \end{aligned}$$

- 2 Use MS_E in calculations

$$\begin{aligned} \text{Var}(\bar{y}_{.k} - \bar{y}_{.k'}) &= \text{Var}(C_k - C_{k'} + \bar{\epsilon}_{.k} - \bar{\epsilon}_{.k'}) \\ &= 2\sigma^2/ab \end{aligned}$$

- 3 Use MS_E in calculations

$$\begin{aligned} \text{Var}(\bar{y}_{i.k} - \bar{y}_{i'.k'}) &= \text{Var}(C_k - C_{k'} + AC_{ik} - AC_{i'k'} + \bar{\epsilon}_{i.k} - \bar{\epsilon}_{i'.k'}) \\ &= 2\sigma^2/b \end{aligned}$$

- 4 Use linear combination $(c-1)MS_E + MS_{AB}$

$$\begin{aligned} \text{Var}(\bar{y}_{i.k} - \bar{y}_{i'.k}) &= \text{Var}(A_i - A_{i'} + \overline{AB}_{i.} - \overline{AB}_{i'..} + \\ &\quad AC_{ik} - AC_{i'k} + \bar{\epsilon}_{i.k} - \bar{\epsilon}_{i'.k}) \\ &= 2(\sigma_{AB}^2/b + \sigma^2/b) \end{aligned}$$

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Unpooled

$$y_{ijk} = \mu + B_j + A_i + AB_{ij} + C_k + AC_{ik} + BC_{jk} + \epsilon_{ijk}$$

- 1 Same: Use MS_{AB} in calculations

- 2 Use MS_{BC} in calculations

$$\begin{aligned} \text{Var}(\bar{y}_{.k} - \bar{y}_{.k'}) &= \text{Var}(C_k - C_{k'} + \overline{BC}_{.k} - \overline{BC}_{.k'} + \\ &\quad \bar{\epsilon}_{.k} - \bar{\epsilon}_{.k'}) \\ &= 2(\sigma_{BC}^2/b + \sigma^2/ab) \end{aligned}$$

- 3 Use linear combination $(a-1)MS_E + MS_{BC}$

$$\begin{aligned} \text{Var}(\bar{y}_{i.k} - \bar{y}_{i'.k'}) &= \text{Var}(C_k - C_{k'} + AC_{ik} - AC_{i'k'} + \overline{BC}_{.k} - \overline{BC}_{.k'} + \\ &\quad \bar{\epsilon}_{i.k} - \bar{\epsilon}_{i'.k'}) \\ &= 2(\sigma_{BC}^2/b + \sigma^2/b) \end{aligned}$$

- 4 Same as before

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