# Balanced Incomplete Block Design

Design of Experiments - Montgomery
Section 4-4



## Balanced Incomplete Block

- Incomplete: cannot fit all trts in each block
- Balanced: each pair of trts occur together $\lambda$ times
- Balanced: Var$(\hat{\tau}_i - \hat{\tau}_j)$ is constant

$a$ trts, $b$ blocks, $r$ replicates, and $k$ trts per block
Total number of obs is $kb = ar = N$

So trt $i$ occurs in $r$ blocks. To have balance, each other trt is equally likely to be with trt $i$ in a block. Since there are $k-1$ other units in a block and $a-1$ other trts, the number of times each pair occurs together is

$$\lambda = r(k-1)/(a-1)$$

where $\lambda$ is an integer. One way to generate this is

- Select $\binom{a}{k}$ blocks and assign each a diff $k$ trt combination
- The number of replicates is $r = \binom{a-1}{k-1}$
- $\lambda = \binom{a-2}{k-2}$
- Sometimes can do this in less than $\binom{a}{k}$ blocks

Extensive list of BIB designs found in Fisher and Yates (1963) and Cochran and Cox (1957)



## Examples

$a = 3, b = 3, k = 2 \rightarrow r = 2, \lambda = 1$

BLOCK

| 1 | 2 | 3 |
|---|---|---|
| A | B | A |
| B | C | C |

$a = 4, k = 2, b = 6 \rightarrow r = 3, \lambda = 1$

BLOCK

| 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| A | A | A | B | B | C |
| B | C | D | C | D | D |

$a = 4, k = 3, b = 4 \rightarrow r = 3, \lambda = 2$

BLOCK

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| A | A | A | B |
| B | B | C | C |
| C | D | D | D |



## Balanced Incomplete Block

- Similar construction as RCBD
- Statistical Model

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- Not all $y_{ij}$ exist because of incompleteness
- Additive effect due to block / No Interaction
- Usual treatment and block restrictions

$$\sum \tau_i = 0 \qquad \sum \beta_j = 0$$

- Nonorthogonality of treatments and blocks

**Use Type III Sums of Squares and lsmeans**

## Analysis of Variance Table

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Blocks | $SS_{Block}$ | $b-1$ | $MS_{Block}$ | |
| Treatment | $SS_{Treatment}$ | $a-1$ | $MS_{Treatment}$ | $F_0$ |
| Error | $SS_E$ | $N-a-b+1$ | $MS_E$ | |
| Total | $SS_T$ | $N-1$ | | |

- $SS_T = \sum\sum y_{ij}^2 - y_{..}^2/N$
- $SS_{Block} = \frac{1}{k}\sum y_{.j}^2 - y_{..}^2/N$
- $SS_{Treatments}$ needs adjustment for incompleteness

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij}y_{.j} \quad \text{where} \quad n_{ij} = \begin{cases} 1 & \text{if trt } i \text{ in blk } j \\ 0 & \text{otherwise} \end{cases}$$

trt $i$'s **total** minus trt $i$'s blk averages

$\sum Q_i = 0$

Cannot consider $n$ equals $r$ ($Q_i$'s correlated)

Can consider $n = \lambda a/k < r$

- $SS_{Treatment(adjusted)} = k\sum Q_i^2/\lambda a = n\sum \hat{\tau}_i^2$
- If $F_0 > F_{\alpha,a-1,N-a-b+1}$ then reject $H_0$



## Model Estimates

- Design matrix $X$ is RCBD with certain rows missing
- Can form normal equations to solve for $\hat{\mu}$, etc.

$$\begin{aligned}\hat{\mu} &= y_{..}/N \\ \hat{\tau}_i &= kQ_i/\lambda a \\ \hat{\beta}_j &= rQ_j'/\lambda b\end{aligned}$$

where

$$Q_i = y_{i.} - \frac{1}{k}\sum n_{ij}y_{.j}$$
$$Q_j' = y_{.j} - \frac{1}{r}\sum n_{ij}y_{i.}$$

$$\begin{aligned}\text{Var}(Q_i) &= \text{Var}(y_{i.}) + \text{Var}\left(\frac{1}{k}\sum n_{ij}y_{.j}\right) - 2\text{Cov}\left(y_{i.}, \frac{1}{k}\sum n_{ij}y_{ij}\right) \\ &= r\sigma^2 + \frac{r}{k^2}k\sigma^2 - \frac{2}{k}r\sigma^2 \\ &= \frac{(k-1)r}{k}\sigma^2 \\ \text{Var}(\hat{\tau}_i) &= \left(\frac{k}{\lambda a}\right)^2 \text{Var}(Q_i) \\ &= \left(\frac{k}{\lambda a}\right)^2 \frac{(k-1)r}{k}\sigma^2 \\ &= \frac{k(a-1)}{\lambda a^2}\sigma^2\end{aligned}$$



## Power and Multiple Comparisons

- Power Calculations
  - Assume $a$ and $k$ are known
  - Limited to values of $b$ such that $\lambda$ an integer

    Non-centrality parameter $\delta = \lambda a\sum \tau_i^2/k\sigma^2$
    Use integer values of $\lambda$ and solve for $b$ to get df

  - Can also use confidence interval estimation method

    Can show Var($\hat{\tau}_i - \hat{\tau}_j$)=$2k\sigma^2/\lambda a$
    Want $\alpha$ level CI to be no larger than $2D$
    $t_{\frac{\alpha}{2},N-a-b+1}\sqrt{2k\sigma^2/a\lambda} = D$

- Multiple Comparisons and Contrasts
  - Similar procedures as ANCOVA
  - Must compute adjusted means (lsmeans)
  - Adjusted mean is $\hat{\mu} + \hat{\tau}_i$
  - Standard error of adjusted mean is $\sqrt{\sigma^2(\frac{k(a-1)}{\lambda a^2} + \frac{1}{N})}$
  - Contrasts based on adjusted treatment totals



## SAS Example

```
options nocenter ps=60 ls=75;
/* From Table 4-22 /*
data example;
 input trt block resp @@;
 cards;
 1 1 73 1 2 74 1 4 71 2 2 75 2 3 67 2 4 72
 3 1 73 3 2 75 3 3 68 4 1 75 4 3 72 4 4 75
;

proc glm;
class block trt;
model resp = block trt;
lsmeans trt / tdiff pdiff adjust=bon stderr;
contrast 'a' trt 1 -1 0 0;
estimate 'b' trt 0 0 1 -1;
---------------------------------------------------------------------
                              Sum of           Mean
Source              DF       Squares         Square  F Value    Pr > F
Model                6    77.7500000     12.9583333    19.94    0.0024
Error                5     3.2500000      0.6500000
Corrected Total     11    81.0000000

Source              DF     Type I SS    Mean Square  F Value    Pr > F
BLOCK                3    55.0000000     18.3333333    28.21    0.0015
TRT                  3    22.7500000      7.5833333    11.67    0.0107

Source              DF   Type III SS    Mean Square  F Value    Pr > F
BLOCK                3    66.0833333     22.0277778    33.89    0.0010
TRT                  3    22.7500000      7.5833333    11.67    0.0107
```

# SAS Example (cont)

```
TRT          RESP      Std Err     Pr > |T|   LSMEAN
           LSMEAN       LSMEAN   H0:LSMEAN=0   Number
1      71.3750000    0.4868051        0.0001     1
2      71.6250000    0.4868051        0.0001     2
3      72.0000000    0.4868051        0.0001     3
4      75.0000000    0.4868051        0.0001     4

Least Squares Means : Adjustment for multiple comparisons: Bonferroni
T for H0: LSMEAN(i)=LSMEAN(j) / Pr > |T|

   i/j       1         2         3         4
   1     .       -0.35806  -0.89514  -5.19183
                   1.0000    1.0000    0.0209
   2  0.358057      .      -0.53709  -4.83378
        1.0000               1.0000    0.0284
   3  0.895144  0.537086      .      -4.29669
        1.0000    1.0000               0.0464
   4  5.191833  4.833775  4.296689      .
        0.0209    0.0284    0.0464

Contrast           DF    Contrast SS   Mean Square  F Value    Pr > F
a                   1      0.0833333     0.0833333     0.13    0.7349

***  Contrast a compares trt1 and trt2.  .13=(-0.35806)*(-0.35806)
***  Different p-values because Bonferroni used in first comparison

                               T for H0:    Pr > |T|   Std Error of
Parameter          Estimate    Parameter=0               Estimate
b               -3.00000000          -4.30     0.0077     0.69821200
```



# Interblock Analysis

Fixed effects analysis is known as intrablock analysis. If blocks are random, we can obtain additional information about $\tau$'s by considering the information between block totals. Based on our model, we can write the block totals as

$$y_{.j} = k\mu + \sum n_{ij}\tau_i + \text{error}$$

and compute the least squares estimates of $\mu$ and $\tau_i$

- The least square estimate for $\tau_i$ is
$$\tilde{\tau}_i = \frac{\sum n_{ij} y_{.j} - kr\overline{y}_{..}}{r - \lambda}$$
- Two estimates are uncorrelated $\rightarrow$ Cov($\hat{\tau}_i, \tilde{\tau}_i$)=0
- Use weighted combination of estimates
- Weights based on the variances of the two estimates

$$\hat{\sigma}^2 = \mathrm{M}S_{\mathrm{E}} \text{ and } \hat{\sigma}^2_\beta = \frac{(b-1)(\mathrm{M}S_{\mathrm{Blks(adj)-MS_E}})}{a(r-1)}$$

$$\tau_i^\star = \begin{cases} \frac{kQ_i(\hat{\sigma}^2 + k\hat{\sigma}^2_\beta) + \left(\sum n_{ij} y_{.j} - kr\overline{y}_{..}\right)\hat{\sigma}^2}{(r-\lambda)\hat{\sigma}^2 + a\lambda(\hat{\sigma}^2 + k\hat{\sigma}^2_\beta)} & \text{if } \hat{\sigma}_\beta > 0 \\ \frac{y_{i.} - (1/a)y_{..}}{r} & \text{if } \hat{\sigma}_\beta = 0 \end{cases}$$



# Interblock Analysis

- Proc Mixed computes combined estimates
- More information used $\rightarrow$ More precise estimates
- Not much benefit if $\sigma^2_\beta >> \sigma^2$
- Can be worse in certain situations
- Combined estimate does not account for uncertainty in variances estimates
- Kenward-Rogers df correction also adjusts std error
- SAS commands:

```
proc mixed;
 class block trt;
 model resp=trt / ddfm=kr;
 random block;
 lsmeans trt / diff adjust=bon;
 contrast 'a' trt 1 -1 0 0;
 estimate 'b' trt 0 0 1 -1;
run;
```



```
The Mixed Procedure
            Dimensions
Covariance Parameters             2
Columns in X                      5
Columns in Z                      4
Subjects                          1
Max Obs Per Subject              12
Observations Used                12
Observations Not Used             0
Total Observations               12

                  Model Information
Data Set                     WORK.EXAMPLE
Covariance Structure         Variance Components
Estimation Method            REML
Residual Variance Method     Profile
Fixed Effects SE Method      Prasad-Rao-Jeske-
                             Kackar-Harville
Degrees of Freedom Method    Kenward-Roger

                     Iteration History
Iteration    Evaluations    -2 Res Log Like       Criterion
        0              1        44.37333968
        1              1        34.22046396      0.00000000
                   Convergence criteria met.

Covariance Parameter Estimates
Cov Parm     Estimate
block          8.0167
Residual       0.6500

           Fit Statistics
-2 Res Log Likelihood            34.2
AIC (smaller is better)          38.2
AICC (smaller is better)         40.6
BIC (smaller is better)          37.0
```

Type 3 Tests of Fixed Effects

| Effect | Num DF | Den DF | F Value | Pr > F |
|---|---|---|---|---|
| trt | 3 | 5.03 | 11.33 | 0.0112 |

Estimates

| Label | Estimate | Std Error | DF | t Value | Pr > \|t\| |
|---|---|---|---|---|---|
| b | -2.9705 | 0.6995 | 5.03 | -4.25 | 0.0080 |

Contrasts

| Label | Num DF | Den DF | F Value | Pr > F |
|---|---|---|---|---|
| a | 1 | 5.03 | 0.08 | 0.7829 |

Least Squares Means

| Effect | trt | Estimate | Std Error | DF | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|
| trt | 1 | 71.4131 | 1.4973 | 3.51 | 47.70 | <.0001 |
| trt | 2 | 71.6164 | 1.4973 | 3.51 | 47.83 | <.0001 |
| trt | 3 | 72.0000 | 1.4973 | 3.51 | 48.09 | <.0001 |
| trt | 4 | 74.9705 | 1.4973 | 3.51 | 50.07 | <.0001 |

Differences of Least Squares Means

| Effect | trt | _trt | Estimate | Std Error | DF | t Value | Pr > \|t\| |
|---|---|---|---|---|---|---|---|
| trt | 1 | 2 | -0.2033 | 0.6995 | 5.03 | -0.29 | 0.7829 |
| trt | 1 | 3 | -0.5869 | 0.6995 | 5.03 | -0.84 | 0.4395 |
| trt | 1 | 4 | -3.5574 | 0.6995 | 5.03 | -5.09 | 0.0037 |
| trt | 2 | 3 | -0.3836 | 0.6995 | 5.03 | -0.55 | 0.6069 |
| trt | 2 | 4 | -3.3541 | 0.6995 | 5.03 | -4.79 | 0.0048 |
| trt | 3 | 4 | -2.9705 | 0.6995 | 5.03 | -4.25 | 0.0080 |

Differences of Least Squares Means

| Effect | trt | _trt | Adjustment | Adj P |
|---|---|---|---|---|
| trt | 1 | 2 | Bonferroni | 1.0000 |
| trt | 1 | 3 | Bonferroni | 1.0000 |
| trt | 1 | 4 | Bonferroni | 0.0225 |
| trt | 2 | 3 | Bonferroni | 1.0000 |
| trt | 2 | 4 | Bonferroni | 0.0289 |
| trt | 3 | 4 | Bonferroni | 0.0480 |



# Other Incomplete Designs

- Youden Square

  Latin Square with one row (col) deleted

  Each trt occurs same number of times in each row (col)

  Columns (rows) for BIBD

  Analysis combination of Latin Square and BIBD

- Partially Balanced Incomplete Block Design

  Doesn't require each pair to occur together $\lambda$ times

  Pair in associate class $i$ appears together $\lambda_i$ times

  All treatments have same # of $i$th associates

  Plus additional restrictions on # of associates

  Extensive list in Bose, Clatworthy, and Shrikhande (1954)

Blocks

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 3 | 4 | 5 | 6 | 1 |
| 4 | 5 | 6 | 1 | 2 | 3 |
| 3 | 4 | 5 | 6 | 1 | 2 |



- Cyclic Designs

  Includes some BIB and PBIB designs

  Consider situation where $r = mk$ and $b = ma$

  Determine $m$ initial blocks, generate others by cycling

  PBIB example is cyclic design with initial block (1243)

  If $k = a$ and rows also blocks get Latin square

- Square, Cubic, and Rectangular Lattices

  Square : $a = k^2$; Cubic : $a = k^3$; Rect: $a = k(k + 1)$

  Square Example : Consider 9 trts and blocks of size three

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

| | | |
|---|---|---|
| A | B | C |
| B | C | A |
| C | A | B |

  Rep 1 blocks : (123)(456)(789) Using Rows

  Rep 2 blocks : (147)(258)(369) Using Columns

  Rep 3 blocks : (168)(249)(357) Using Latin Square

  Additional Reps obtained from orthogonal squares